Study Guide on Credibility with Shifting Risk Parameters for the Society of Actuaries (SOA) Exam GIADV: Advanced Topics in General Insurance

(Based on Stuart Klugman's Paper, *"Credibility with Shifting Risk Parameters"***)**

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Source: Klugman, S.A., "Credibility with Shifting Risk Parameters", SOA Study Note, 2014.

Problem CSRP-1. When there is a sequence of observations taken over several time periods, and the goal is to forecast the next observation in the series, what two simplifying assumptions are made in Klugman's paper? (Klugman, p. 2)

Solution CSRP-1. The following simplifying assumptions are made:

(i) All external influences (e.g., inflationary trend and business cycles) have been removed.
(ii) The amount of data that produced each observation is relatively small.

Problem CSRP-2. If independence among all observations and equal sample sizes are assumed, then what becomes the best (minimum variance) unbiased linear estimator of the true mean? (Klugman, p. 3)

Solution CSRP-2. The **sample mean** becomes the minimum variance unbiased linear estimator of the true mean under these assumptions.

Problem CSRP-3. ***Fill in the blanks***: If it is thought that observations are dependent and that an AR(1) process is appropriate for describing them, then the best prediction of the next observation is a weighted average of _________ and __________, using __________ as the weight. (Klugman, p.3)

Solution CSRP-3. If it is thought that observations are dependent and that an AR(1) process is appropriate for describing them, then the best prediction of the next observation is a weighted average of **the most recent observation** and **the sample mean**, using **the sample lag one autocorrelation** as the weight.

Problem CSRP-4. Suppose you have the following observations: 123, 120, 325, 345, 354.
Using an AR(1) process, the lag 1 autocorrelation is 0.404151.
Find the best prediction of the sixth observation using this AR(1) process.

Solution CSRP-4. The sample mean is (123+120+325+345+354)/5 = 253.4.

We take a weighted average of this sample mean, 253.4, and the most recent observation, 354. The weight assigned to the most recent observation of 354 is the lag 1 autocorrelation of 0.404151.

Predicted sixth observation = 354*0.404151 + 253.4*(1-0.404151) = **294.0576**.

Problem CSRP-5.
(a) By what assumption is the time-series approach motivated?
(b) As an implication of this assumption, what is the only factor on which correlation depends?
(c) What is another way of looking at this assumption?
(Klugman, p.3)

Solution CSRP-5.
(a) The time-series approach is motivated by an assumption that **observations are correlated**.
(b) Time between the observations is the only factor on which correlation depends.
(c) Another way of looking at this is that **the means are changing over time**. (Klugman, p.3)

Problem CSRP-6. When the sample size is small, Klugman (p. 3) suggests that a credibility approach may be appropriate. By what two factors is the credibility approach motivated?

Solution CSRP-6.
1. With limited data, any forecast based only on that data may have considerable variability.
2. There may be multiple forecasts (such as for other insured groups), and, according to credibility theory**,** in aggregate, multiple forecasts are more accurate when individual means are credibility-weighted with an overall mean. (Klugman, p.3)

Problem CSRP-7. Klugman's "Credibility with Shifting Risk Parameters" demonstrates a method of combining *which two approaches*? (Klugman, p.3)

Solution CSRP-7. Klugman's "Credibility with Shifting Risk Parameters" demonstrates a method of combining **time-series** and **credibility** approaches.

Problem CSRP-8. Within Klugman's discussion, what three types of changes does the phrase "change over time" *not* encompass? What should be done with these types of changes *before* a time-series or credibility approach is applied? (Klugman, p. 4)

Solution CSRP-8. Klugman's use of the term "change over time does *not* encompass:
(i) Trend**;**
(ii) Abrupt changes at specific time points (e.g., changes in marketing, underwriting, regulation);
(iii) Economic cycles.
These types of changes should be accounted for with data adjustments before a time-series or credibility approach is applied. (Klugman, p. 4)

Problem CSRP-9. If the number of exposures for each observation is allowed to differ (instead of all exposures being assumed to be equal to the same quantity, such as 1), what is the only estimate that would be affected by this difference in assumptions? (Klugman, p. 4)

Solution CSRP-9. The **process variance estimate** is the only one that would be affected if the number of exposures for each observation is allowed to differ.

Problem CSRP-10. Klugman (p. 4) notes that autocorrelations have a high standard error. What is a rough approximation of the standard error for an autocorrelation?

Solution CSRP-10. A rough approximation of the standard error for an autocorrelation is **the square root of the ratio of 1 and the sample size.**

Problem CSRP-11. Suppose you have the following observations: 123, 120, 325, 345, 354.

Using an AR(1) process, the lag 1 autocorrelation is 0.404151.

Use Klugman's "rough approximation" (p. 4) for the standard error of this autocorrelation.

Solution CSRP-11. The rough approximation is to take the square root of the ratio of 1 and the sample size. Here, the sample size is 5, so the rough approximation for standard error is $\sqrt{(1/5)}$ = **0.447214**.

Problem CSRP-12. What does it mean to say that *observations have been transformed to become weakly stationary*? Provide two specific implications. (Klugman, p. 5)

Solution CSRP-12.
1. The unconditional mean is the same at all times.
2. The covariance between two observations depends only on the time that separates them.

Problem CSRP-13. If observations have been transformed to become weakly stationary and you are looking at a sequence of observations, based on the first and second moments, what is it *not* possible to identify? (Klugman, p. 5)

Solution CSRP-13. It is not possible to identify **the time period from which the observations originated.**

Problem CSRP-14. For the autoregressive model of order 1 – AR(1) – you are given the following:

• μ is the sample mean.
• $-1 < \rho < 1$ is the lag 1 autocorrelation coefficient.
• $\{\varepsilon_t\}$ is a sequence of independent random variables with a mean of zero and a standard deviation of σ_ε.
• x_t is the observation at time t.

(a) Provide the formula by which observation x_t is related to the previous observation x_{t-1}.

(b) Provide a formula for the variance of x_t, $Var(x_t) = \sigma_x^2$.
(c) Provide a formula for the correlation coefficient $Corr(x_{t-s}, x_t)$ between observations x_{t-s} and x_t.

(Klugman, p. 5)

Solution CSRP-14.
(a) $\mathbf{x_t = \rho^*x_{t-1} + (1 - \rho)^*\mu + \varepsilon_t}$.
(b) $\mathbf{Var(x_t) = \sigma_x^2 = \sigma_\varepsilon^2/(1 - \rho^2)}$.
(c) $\mathbf{Corr(x_{t-s}, x_t) = \rho^s}$.

Problem CSRP-15. Suppose you have the following observations: 123, 120, 325, 345, 354.

Using an AR(1) process, the lag 1 autocorrelation is 0.404151.

What is the correlation coefficient $Corr(x_2, x_5)$ between the second and fifth observations?

Solution CSRP-15. We use the formula $\mathbf{Corr(x_{t-s}, x_t) = \rho^s}$**,** where t = 5, s = 3 (since t – s = 5 – s = 2, and so s must be 3), and ρ = 0.404151. Our answer is thus $Corr(x_2, x_5) = 0.404151^3$ = **0.066013**.

Problem CSRP-16. According to Klugman (p. 5), what is an *ad hoc* way to estimate the standard deviation of σ_ε for a sample to which the AR(1) model is applied? Provide a formula to use in this estimation.

Solution CSRP-16. The sample standard deviation of the residuals, $\mathbf{\varepsilon^{\wedge}_t = x_t - \rho^*x_{t-1} - (1-\rho)^*\ \mu}$, can be used to estimate σ_ε. In this formula, ρ and μ are the *sample* lag 1 autocorrelation coefficient and mean, respectively.

Problem CSRP-17. Suppose you have the following observations: 123, 120, 325, 345, 354.

Using an AR(1) process, the lag 1 autocorrelation is 0.404151.

Estimate the standard deviation σ_ε for the sequence of independent random variables $\{\varepsilon_t\}$ associated with this sample.

Solution CSRP-17. We want to find the sample standard deviation of the residuals $\varepsilon^{\wedge}_t = x_t - \rho^*x_{t-1} - (1-\rho)^*\ \mu$ for each t.

We are given that ρ = 0.404151.
We also calculate the sample mean μ = (123+120+325+345+354)/5 = 253.4.

Thus, for each t, $\varepsilon^{\wedge}_t = x_t - 0.404151^*x_{t-1} - (1-0.404151)^*253.4$;
$\varepsilon^{\wedge}_t = x_t - 0.404151^*\ x_{t-1} - 150.9881$.
We have the following calculations (over t = 2 to 5, since x_{t-1} does not exist for t = 1).

t	x_t	x_{t-1}	$\hat{\varepsilon}_t$
2	120	123	-80.698673
3	325	120	125.51378
4	345	325	62.662825
5	354	345	63.579805

The mean residual amount is (-80.698673 + 125.51378 + 62.662825 + 63.579805)/4 = 42.76443425.

The sample variance is the sum of the squares of the ($\hat{\varepsilon}_t$ – Mean Residual) divided by the sample size minus 1 (here, 5-1 = 4):

$\sigma_\varepsilon^2 = [(-80.698673 - 42.76443425)^2 + (125.51378 - 42.76443425)^2 + (62.662825 - 42.76443425)^2 + (63.579805 - 42.76443425)^2]/4 = 5729.954672$.
It follows that our estimate of $\sigma_\varepsilon = \sqrt{}(5729.954672)$ = **75.69646407**.

Problem CSRP-18. Suppose you have the following observations: 123, 120, 325, 345, 354.

Using an AR(1) process, the lag 1 autocorrelation is 0.404151.

Estimate the variance of x_t, $\mathrm{Var}(x_t) = \sigma_x^2$.

Solution CSRP-18. From Solution CSRP-14(b), we have the formula
$\mathrm{Var}(x_t) = \sigma_x^2 = \sigma_\varepsilon^2/(1 - \rho^2)$.

From Solution CSRP-17, we have the estimate of $\sigma_\varepsilon = 86.94108067$ and $\sigma_\varepsilon^2 = 5729.954672$.
We are also given $\rho = 0.404151$.

Therefore, $\sigma_x^2 = 5729.954672/(1 - 0.404151^2) = \boldsymbol{\sigma_x^2 = 6848.589852}$.

Problem CSRP-19. Apart from the approach of using the sample mean, lag 1 autocorrelation coefficient, and residuals to calculate the standard error, what two other approaches are possible, according to Klugman (pp. 5 – 6)?

Solution CSRP-19.
1. Set up a regression formula for the AR(1) model and obtain a least-squares estimate and its standard error.
2. Make a distributional assumption and derive a maximum likelihood estimate and its standard error.

Problem CSRP-20. ***Fill in the blanks*** (Klugman, p. 6): To forecast future observations using the AR(1) model, one can apply the AR(1) model formula with the error term for future values set equal to _______, which is the error term's ___________.

Solution CSRP-20. To forecast future observations using the AR(1) model, one can apply the AR(1) model formula with the error term for future values set equal to **zero**, which is the error term's **expected value**.

Problem CSRP-21. Which source of variability does the forecast error of the AR(1) model, as calculated using the methodology presented in Klugman's paper, reflect? Which source does it *not* reflect? (Klugman, p. 6)

Solution CSRP-21. The forecast model **reflects process variability**, but **does not reflect estimation error**.
Problem CSRP-22. Under what circumstance is an MA(1) model appropriate? (Klugman, p. 6)

Solution CSRP-22. An MA(1) model is appropriate when *only* the lag 1 autocorrelation is nonzero.

Problem CSRP-23. For an MA(1) model, you are given the following:

- μ is the sample mean.
- There is a parameter θ, such that $-1 < \theta < 1$.
- $\{\varepsilon_t\}$ is a sequence of independent random variables with a mean of zero and a standard deviation of σ_ε.
- x_t is the observation at time t.

(a) Provide the formula for the observation x_t.
(b) Provide a formula for the variance of x_t, $Var(x_t) = \sigma_x^2$.
(c) Provide a formula for $Cov(x_{t-1}, x_t)$, the covariance between two adjacent observations x_{t-1} and x_t.
(d) Provide a formula for the correlation coefficient $Corr(x_{t-1}, x_t)$ between observations x_{t-1} and x_t.

(Klugman, p. 6)

Solution CSRP-23.
(a) $\mathbf{x_t = \mu + \varepsilon_t - \theta*\varepsilon_{t-1}}$**.**
(b) $\mathbf{Var(x_t) = \sigma_x^2 = (1 + \theta^2)\sigma_\varepsilon^2}$**.**
(c) $\mathbf{Cov(x_{t-1}, x_t) = -\theta\sigma_\varepsilon^2}$**.**
(d) $\mathbf{Corr(x_{t-1}, x_t) = -\theta/(1 + \theta^2)}$**.**

Problem CSRP-24. Suppose you have the following observations: 123, 120, 325, 345, 354.

The lag 1 autocorrelation is 0.404151.

If you assume that an MA(1) process is appropriate to use as a model, calculate the value of the parameter θ.

Solution CSRP-24. We use the formula $\mathrm{Corr}(x_{t-1}, x_t) = -\theta/(1 + \theta^2)$.
We are given that $\mathrm{Corr}(x_{t-1}, x_t)$, the lag 1 autocorrelation, is 0.404151.
Therefore, $0.404151 = -\theta/(1 + \theta^2) \rightarrow$
$0.404151*(1 + \theta^2) = -\theta \rightarrow$
$0.404151\theta^2 + \theta + 0.404151 = 0$, which is a quadratic equation.
The solutions to the quadratic equation are −0.5087602145798299 and −1.965562501434729, but only one of these solutions is within the constraint that $-1 < \theta < 1$.
Therefore, $\boldsymbol{\theta = -0.5087602145798299}$.

Problem CSRP-25. Klugman (p. 7) states that any MA model can be rewritten as an AR model, but with an infinite number of terms. Provide the first six terms of the infinite expression for x_t, the observation at time t, and then end the expression with "… + ε_t", where ε_t is the residual.

Use the following notation:
• μ is the sample mean.
• There is a parameter θ, such that $-1 < \theta < 1$.

Solution CSRP-25.
$$\mathbf{x_t = \mu - \theta(x_{t-1} - \mu) - \theta^2(x_{t-2} - \mu) - \theta^3(x_{t-3} - \mu) - \theta^4(x_{t-4} - \mu) - \theta^5(x_{t-5} - \mu) \ldots + \varepsilon_t}$$

Problem CSRP-26. Suppose you have the following observations: 123, 120, 325, 345, 354.

The lag 1 autocorrelation is 0.404151.

If you assume that an MA(1) process is appropriate to use as a model, find the predicted sixth observation.

Solution CSRP-26. We use the formula for the infinite expansion for x_t:
$$\mathbf{x_t = \mu - \theta(x_{t-1} - \mu) - \theta^2(x_{t-2} - \mu) - \theta^3(x_{t-3} - \mu) - \theta^4(x_{t-4} - \mu) - \theta^5(x_{t-5} - \mu) \ldots + \varepsilon_t}$$

Here, we seek to find x_6. Because there are only five observations given, the infinite expansion can be reduced to a finite one, and the expected value of the residual ε_t will be 0.

Thus, we need to find $\mathbf{x_6 = \mu - \theta(x_5 - \mu) - \theta^2(x_4 - \mu) - \theta^3(x_3 - \mu) - \theta^4(x_2 - \mu) - \theta^5(x_1 - \mu).}$

In Solution CSRP-24, we found that $\theta = -0.5087602145798299$.

We also calculate the sample mean $\mu = (123+120+325+345+354)/5 = 253.4$.

Our observations are $x_1 = 123$, $x_2 = 120$, $x_3 = 325$, $x_4 = 345$, and $x_5 = 354$.

Thus, we solve for $x_6 = 253.4 - (-0.5087602145798299)(354 - 253.4) - (-0.5087602145798299)^2(345 - 253.4) - (-0.5087602145798299)^3(325 - 253.4) - (-0.5087602145798299)^4(120 - 253.4) - (-0.5087602145798299)^5(123 - 253.4) =$
Predicted sixth observation = $\mathbf{x_6}$ = 294.7931598.

Problem CSRP-27.
(a) If an MA(1) model is applied not to the observations x_t, but to the *differences of the observations*, $y_t = x_t - x_{t-1}$, with no constant term, what are two names for the resulting model?
(b) What is the expression for $y_t = x_t - x_{t-1}$ in this model?
You are given that there is a parameter θ, such that $-1 < \theta < 1$. Also use ε_t as the residual at time t.
(c) If one expresses this model as a pure AR model, an infinite expansion for x_t is possible. Provide the first five terms of this expansion, in addition to the final term "+ ε_t", where ε_t is the residual. (Klugman, p. 7)

Solution CSRP-27.
(a) This is the **ARIMA(0, 1, 1) model**, also referred to as a **simple exponential smoothing model**.
(b) $y_t = x_t - x_{t-1} = \varepsilon_t - \theta*\varepsilon_{t-1}$.
(c) $x_t = (1-\theta)*[x_{t-1} + \theta x_{t-2} + \theta^2 x_{t-3} + \theta^3 x_{t-4} + \theta^4 x_{t-5} + \ldots] + \varepsilon_t$.

Problem CSRP-28. Suppose you have the following observations: 123, 120, 325, 345, 354.

The lag 1 autocorrelation is 0.404151.

If you assume that an ARIMA(0, 1, 1) model with parameter θ = −0.5087602145798299 is appropriate, find the predicted sixth observation.

Solution CSRP-28. We use the formula for the infinite expansion for x_t:
$x_t = (1-\theta)*[x_{t-1} + \theta x_{t-2} + \theta^2 x_{t-3} + \theta^3 x_{t-4} + \theta^4 x_{t-5} + \ldots] + \varepsilon_t$.

Here, we seek to find x_6. Because there are only five observations given, the infinite expansion can be reduced to a finite one, and the expected value of the residual ε_t will be 0.

Our observations are $x_1 = 123$, $x_2 = 120$, $x_3 = 325$, $x_4 = 345$, and $x_5 = 354$.
Thus, we need to find $x_6 = (1-\theta)*[x_5 + \theta x_4 + \theta^2 x_3 + \theta^3 x_2 + \theta^4 x_1] =$
$(1-(-0.5087602145798299))*[354 + (-0.5087602145798299)*345 +$
$(-0.5087602145798299)^2*325 + (-0.5087602145798299)^3*120 +$
$(-0.5087602145798299)^4*123] =$
Predicted sixth observation = x_6 = 384.7911896.

Problem CSRP-29. When using an empirical Bayes approach as described by Klugman (p. 8), you are given the following notation:

- $X_{i,t}$ = random observation from group i at time t
- k = number of groups (a group is the entity for which we want to make a forecast)
- n = number of time periods for each group
- ξ_i = the mean for group i, an unobservable random quantity
- $\mu = E(\xi_i)$
- $\tau^2 = Var(\xi_i)$
- $\sigma^2 = Var(X_{i,t} \mid \xi_i)$

(a) According to Klugman, what is the objective of the empirical Bayes approach?
(b) What is the formula for $\ddot{X}_i$, the sample mean of the observations $X_{i,t}$?
(c) The empirical Bayes approach uses credibility-weighting between $\ddot{X}_i$ and μ to arrive at an estimate of ξ_i. What is the formula for the credibility factor Z applied to $\ddot{X}_i$?

Solution CSRP-29.
(a) The objective of the empirical Bayes approach is to **use the available data to estimate the k unknown means**.
(b) $\ddot{X}_i = (1/n)*_{t=1}{}^{n}\Sigma(X_{i,t})$
(c) $Z = n/(n + \sigma^2/\tau^2)$

Problem CSRP-30. When using an empirical Bayes approach as described by Klugman (p. 8), you are given the following notation:

- $X_{i,t}$ = random observation from group i at time t
- k = number of groups (a group is the entity for which we want to make a forecast)
- n = number of time periods for each group
- ξ_i = the mean for group i, an unobservable random quantity
- $\mu = E(\xi_i)$
- $\tau^2 = Var(\xi_i)$
- $\sigma^2 = Var(X_{i,t} \mid \xi_i)$
- $\ddot{X}_i$ = the sample mean of the observations $X_{i,t}$

For the second step of the empirical Bayes approach, three parameters must be estimated from the data.

(a) Give the formula for the estimate μ^ of μ.
(b) Give the formula for the estimate σ^^{2} of σ^2.
(c) Give the formula for the estimate τ^^{2} of τ^2, using μ^ and σ^^{2} within the formula.

Solution CSRP-30.
(a) $\mu\text{^} = (1/kn)\ _{i=1}{}^{k}\Sigma(_{t=1}{}^{n}\Sigma(X_{i,t}))$.
(b) $\sigma\text{^}^2 = (1/[k(n-1)])\ _{i=1}{}^{k}\Sigma(_{t=1}{}^{n}\Sigma(X_{i,t} - \ddot{X}_i)\text{^}2)$.
(c) $\tau\text{^}^2 = (1/(k-1))*\ _{i=1}{}^{k}\Sigma(\ddot{X}_i - \mu\text{^})^2 - (1/n)*\sigma\text{^}^2$.

Problem CSRP-31. You are using an empirical Bayes approach with a data set comprised of 23 groups and 35 time periods of data for each group.

You have estimated parameters μ^ = 1256, σ^^{2} = 11600, and τ^^{2} = 1606. The observed mean of group i, $\ddot{X}_i$, is 1500.

(a) Find the estimated credibility factor Z that would be applicable to $\ddot{X}_i$.
(b) Find the estimate of the mean for group i, $\xi\text{^}_i$.

Solution CSRP-31.
(a) We use the formula $Z = n/(n + \sigma^2/\tau^2)$. Here, we are given n = 35, and our estimates $\sigma^{\wedge 2}$ = 11600 and $\tau^{\wedge 2}$ = 1606. Thus, Z = 35/(35 +11600/1606) = **Z = 0.828933786**.

(b) To find $\xi^\wedge_i$, we apply Z to $\ddot{X}_i$ and the complement of Z to $\mu^\wedge$: $\xi^\wedge_i = Z*\ddot{X}_i + (1-Z)*\mu^\wedge$ = 0.828933786*1500 + (1-0.828933786)*1256 = $\boldsymbol{\xi^\wedge_i}$ **= 1458.259844**.

Problem CSRP-32. For a *random effects linear model,* provide the equation for $X_{i,t}$, the random observation from group i at time t. (Klugman, p. 9)

Solution CSRP-32. $\boldsymbol{X_{i,t} = \mu + \alpha_i + \varepsilon_{i,t}}$, where α_i follows a Normal distribution with mean 0 and variance τ^2, and $\boldsymbol{\varepsilon_{i,t}}$ follows a Normal distribution with mean 0 and variance σ^2.

Problem CSRP-33.
(a) What distinguishes the random effects linear model from a traditional fixed effects model?
(b) If a fixed effects regression model were applied to time-series data, what problem would be encountered? (Klugman, p. 9)

Solution CSRP-33.
(a) The distinction is that the middle term in a random effects linear model (α_i) is a random variable.
(b) If a fixed effects regression model were used, the estimator for each group would be the sample mean, and there would be no credibility applied to individual observations.

Problem CSRP-34. ***Fill in the blanks*** (Klugman, p. 9): Unlike the Bühlmann-Straub approach, the random effects linear model incorporates a __________ assumption and, in particular, there is a correspondence between the assumed ___________ and least-squares estimation.

Solution CSRP-34. Unlike the Bühlmann-Straub approach, the random effects linear model incorporates a **distributional** assumption and, in particular, there is a correspondence between the assumed **Normal distribution** and least-squares estimation.

Problem CSRP-35. Because, for a random effects linear model, the distributions are specified, what approach could be used? (Klugman, p. 9)

Solution CSRP-35. Because the distributions are specified, *maximum likelihood estimation* could be used. (Klugman, p. 9)

Problem CSRP-36. **(a)** Klugman (p. 9) discusses the tendency of variance estimates to be biased. Discuss the specific reason why the variance estimate would be biased with regard to a Normal distribution.
(b) What is the name of the technique that was developed to address this problem?

Solution CSRP-36.
(a) For a sample taken from the Normal distribution, the maximum likelihood estimate of the variance would have n (the number of observations) in the denominator, rather than (n-1), which would be the value that would produce an unbiased estimate.

(b) The name of the technique that was developed to address this problem is **REML – Restricted Maximum Likelihood**.

Problem CSRP-37. Why, according to Klugman (p. 9), is writing the regular loglikelihood function a challenge?

Solution CSRP-37. Writing the regular loglikelihood function is a challenge because the observations within a group are dependent. (Klugman, p. 9)

Problem CSRP-38. You are given a random effects linear model with the following equation for $X_{i,t}$, the random observation from group i at time t: $X_{i,t} = \mu + \alpha_i + \varepsilon_{i,t}$, where α_i follows a Normal distribution with mean 0 and variance τ^2, and $\varepsilon_{i,t}$ follows a Normal distribution with mean 0 and variance σ^2.

With respect to observations in group i, provide expressions for the following, using the terms defined above:
(a) $E(X_{i,t})$, the expected value of $X_{i,t}$
(b) $Var(X_{i,t})$, the variance of $X_{i,t}$
(c) $Cov(X_{i,t}, X_{i,s})$, the covariance of $X_{i,t}$ with another observation $X_{i,s}$ from group i, where $t \neq s$.
(Klugman, p. 10)

Solution CSRP-38.
(a) $\mathbf{E(X_{i,t}) = \mu}$
(b) $\mathbf{Var(X_{i,t}) = \tau^2 + \sigma^2}$
(c) $\mathbf{Cov(X_{i,t}, X_{i,s}) = \tau^2}$

Problem CSRP-39. Under a random effects linear model, Klugman (p. 10) notes that the vector of observations from group I has the multivariate Normal density function

$f(x_{i,1}, \ldots, x_{i,n} \mid \mu, \sigma^2, \tau^2) = (1/[(2\pi)^{n/2}*[\det[\mathbf{V}]]^{1/2}])*\exp(-(1/2)*[\mathbf{x_i}-\boldsymbol{\mu}]^T*\text{inverse}[\mathbf{V}]*[\mathbf{x_i}-\boldsymbol{\mu}])$.

(a) Define the vector $[\mathbf{x_i}]$.
(b) Define the vector $[\boldsymbol{\mu}]$.
(c) Define the matrix $[\mathbf{V}]$.

Solution CSRP-39.
(a) $[\mathbf{x_i}] = (x_{i,1}, \ldots, x_{i,n})$ is a vector of the n observations from group i.
(b) $[\boldsymbol{\mu}] = (\mu, \ldots, \mu)$ is a vector containing n instances of the mean μ.
(c) $[\mathbf{V}]$ is an n by n matrix with $\sigma^2 + \tau^2$ on the diagonal and τ^2 everywhere else.

Problem CSRP-40. ***Fill in the blanks*** (Klugman, p. 10): In a random effects linear model, maximizing the likelihood with respect to the two variance terms results in a formula similar to the __________ formula, but the estimator of τ^2 has a different _________.

Solution CSRP-40. In a random effects linear model, maximizing the likelihood with respect to the two variance terms results in a formula similar to the **empirical Bayes** formula, but the estimator of τ^2 has a different **denominator**. (Klugman, p. 10)

Problem CSRP-41. Klugman (p. 10), provides the regular loglikelihood function using a random effects linear model. This function is

ln L = (-nk/2)*ln(2π) – (k/2)*ln[det[**V**]] – $(1/2)_{i=1}{}^{k}\Sigma([\mathbf{x_i}\text{-}\boldsymbol{\mu}]^T$*inverse[**V**]*[$\mathbf{x_i}$-**μ**]), where k is the number of groups.

Klugman then provides an altered loglikelihood function using the REML estimator. What are the two differences in this function from the function presented above?

Solution CSRP-41.
Difference 1: The mean μ in the vector [**μ**] is replaced by the estimate μ^.
Difference 2: The loglikelihood function ln L has one additional term:
(-1/2)ln[k × sum[inverse[**V**]]], where the "sum" function adds the elements of the matrix or vector to which it is applied.

Problem CSRP-42. ***Fill in the blanks*** (Klugman, p. 10): For a REML loglikelihood function, the basis of the estimator is to obtain a likelihood function that does not depend on the _________. The likelihood is based on the __________ after the subtraction of the estimated mean. The values that maximize this function exactly match the ___________.

Solution CSRP-42. For a REML loglikelihood function, the basis of the estimator is to obtain a likelihood function that does not depend on the **mean**. The likelihood is based on the **residuals** after the subtraction of the estimated mean. The values that maximize this function exactly match the **empirical Bayes estimates**. (Klugman, p. 10)

Problem CSRP-43. What is the necessary condition for REML estimates to exactly match empirical Bayes estimates? (Klugman, p. 10)

Solution CSRP-43. The necessary condition is that total exposures must be identical across groups. When exposures differ among groups, the REML estimates will be biased. (Klugman, p. 10)

Problem CSRP-44. On page 11, Klugman introduces a three-level model that incorporates autocorrelations and allows for both time dependence and credibility. Qualitatively describe each of the three levels.

Solution CSRP-44.
Level 1 states that there is a mean for the group (denoted ξ) that is stable over time.
Level 2 reflects the dependency structure of the means of the groups over time.
Level 3 is the observation itself.

Problem CSRP-45. Let X_1, …., X_n be random variables that represent observations at times 1 through n. Suppose you have credibility weights Z_0, Z_1, …, Z_n, which are used to estimate a future random observation, X_{n+d}, using the following formula:

$X\hat{}_{n+d} = Z_0 + Z_1*X_1 + \ldots + Z_n*X_n$.

Then, according to Klugman (p. 11), the values that minimize the expected square error are solutions to equations for $E[X_{n+d}]$ and $Cov(X_t, X_{n+d})$ for t = 1, …, n. What are these equations?

Solution CSRP-45.
$\mathbf{E[X_{n+d}] = Z_0 + {}_{t=1}{}^{n}\Sigma(Z_t*E(X_t))}$
$\mathbf{Cov(X_t, X_{n+d}) = {}_{s=1}{}^{n}\Sigma(Z_t*Cov(X_t, X_s))}$ for t = 1, …, n.

Problem CSRP-46. In Klugman's three-level model, the following information is given:
• ξ is the mean for the group, which is stable over time and drawn from a distribution with mean μ and variance τ^2.
• The means for the observations at times 1 through n are (θ_1, …, θ_n), drawn from a multivariate distribution where each value has a mean of ξ.
• For any t and s among the values 1, 2, …, n, n+d, let $Cov(\theta_t, \theta_s) = \delta_{t,s}$.
• The value for the observation X_t is drawn for a distribution with mean θ_t and variance σ^2/w_t, where w_t is a known weight.

(a) What general idea does this model convey regarding the mean at time t?
(b) What is the implication if one time period has a large mean?
(Klugman, p. 11)

Solution CSRP-46.
(a) The mean at time t is random, but related to the mean at other times.
(b) If one time period has a large mean, this large mean may persist for several future periods.
(Klugman, p. 11)

Problem CSRP-47. In Klugman's three-level model, the following information is given:
• ξ is the mean for the group, which is stable over time and drawn from a distribution with mean μ and variance τ^2.
• The means for the observations at times 1 through n are (θ_1, …, θ_n), drawn from a multivariate distribution where each value has a mean of ξ.
• For any t and s among the values 1, 2, …, n, n+d, let $Cov(\theta_t, \theta_s) = \delta_{t,s}$.
• The value for the observation X_t is drawn for a distribution with mean θ_t and variance σ^2/w_t, where w_t is a known weight.

Provide expressions for the following, using the terms defined above:
(a) $E(X_t)$, the expected value of X_t
(b) $Var(X_t)$, the variance of X_t
(c) $Cov(X_t, X_s)$, the covariance of X_t with an observation X_s from the same group, where $t \neq s$.
(Klugman, p. 12)

Solution CSRP-47.
(a) $E(X_t) = \mu$
(b) $Var(X_t) = \sigma^2/w_t + \delta_{t,t} + \tau^2$
(c) $Cov(X_t, X_s) = \delta_{t,s} + \tau^2$, for $t \neq s$.

Problem CSRP-48. In Klugman's three-level model, the following information is given:
• ξ is the mean for the group, which is stable over time and drawn from a distribution with mean μ and variance τ^2.
• The means for the observations at times 1 through n are $(\theta_1, \ldots, \theta_n)$, drawn from a multivariate distribution where each value has a mean of ξ.
• For any t and s among the values 1, 2, …, n, n+d, let $Cov(\theta_t, \theta_s) = \delta_{t,s}$.
• The value for the observation X_t is drawn for a distribution with mean θ_t and variance σ^2/w_t, where w_t is a known weight.

You are seeking to solve for the credibility weights $Z_1, \ldots, Z_n$ for the observations $X_1, \ldots, X_n$.

Provide a matrix equation in the following format:

$$\begin{bmatrix} Z_1 \\ Z_2 \\ \ldots \\ Z_n \end{bmatrix} = \text{INVERSE}\left[\begin{array}{|c|c|c|c|} \hline & & \ldots & \\ \hline & & \ldots & \\ \hline \ldots & \ldots & \ldots & \ldots \\ \hline & & \ldots & \\ \hline \end{array}\right] * \begin{bmatrix} \; \\ \; \\ \ldots \\ \; \end{bmatrix}$$

Solution CSRP-48.

$$\begin{bmatrix} Z_1 \\ Z_2 \\ \ldots \\ Z_n \end{bmatrix} = \text{INVERSE}\left[\begin{array}{|c|c|c|c|} \hline \boldsymbol{\sigma^2/w_1 + \delta_{1,1} + \tau^2} & \boldsymbol{\delta_{1,2} + \tau^2} & \ldots & \boldsymbol{\delta_{1,n} + \tau^2} \\ \hline \boldsymbol{\delta_{2,1} + \tau^2} & \boldsymbol{\sigma^2/w_2 + \delta_{2,2} + \tau^2} & \ldots & \boldsymbol{\delta_{2,n} + \tau^2} \\ \hline \ldots & \ldots & \ldots & \ldots \\ \hline \boldsymbol{\delta_{n,1} + \tau^2} & \boldsymbol{\delta_{n,2} + \tau^2} & \ldots & \boldsymbol{\sigma^2/w_n + \delta_{n,n} + \tau^2} \\ \hline \end{array}\right] * \begin{bmatrix} \boldsymbol{\delta_{1,n+d} + \tau^2} \\ \boldsymbol{\delta_{2,n+d} + \tau^2} \\ \ldots \\ \boldsymbol{\delta_{n,n+d} + \tau^2} \end{bmatrix}$$

Problem CSRP-49. Based on Klugman's discussion on page 12, separate the matrix in Solution CSRP-48 above into the following three components:
(a) The diagonal matrix [D];
(b) The covariance matrix [Σ];

(c) The matrix [T].
(d) If the equation in Solution CSRP-48 can be represented as [Z] = inverse[V]*[S], what is the equivalent equation in terms if [Z], [S], [D], [Σ], and [T]?

Solution CSRP-49.
(a) [D] =

σ^2/w_1	0	…	0
0	σ^2/w_2	…	0
…	…	…	…
0	0	…	σ^2/w_n

(b) [Σ] =

$\delta_{1,1}$	$\delta_{1,2}$	…	$\delta_{1,n}$
$\delta_{2,1}$	$\delta_{2,2}$	…	$\delta_{2,n}$
…	…	…	…
$\delta_{n,1}$	$\delta_{n,2}$	…	$\delta_{n,n}$

(c) [T] =

τ^2	τ^2	…	τ^2
τ^2	τ^2	…	τ^2
…	…	…	…
τ^2	τ^2	…	τ^2

(d) [Z] = inverse[[D] + [Σ] + [T]]*[S]

Problem CSRP-50. In Klugman's three-level model, the following information is given:
• ξ is the mean for the group, which is stable over time and drawn from a distribution with mean μ and variance τ^2.
• The means for the observations at times 1 through n are $(\theta_1, \ldots, \theta_n)$, drawn from a multivariate distribution where each value has a mean of ξ.
• For any t and s among the values 1, 2, …, n, n+d, let $\text{Cov}(\theta_t, \theta_s) = \delta_{t,s}$.
• The value for the observation X_t is drawn for a distribution with mean θ_t and variance σ^2/w_t, where w_t is a known weight.
• The vector [Z] of the credibility weights $Z_1, \ldots, Z_n$ for the observations $X_1, \ldots, X_n$ can be obtained as the solution to the matrix equation [Z] = inverse[V]*[S], as expressed in Solution CSRP-48.

(a) What is the expression for mean squared error as a function of the general value of [Z]?
(b) What is the expression for mean squared error as a function of the value of [Z] that minimizes mean squared error?

(Klugman, p. 13)

Solution CSRP-50.
(a) $\boldsymbol{\sigma^2/w_{n+d} + \delta_{n+d,n+d} + \tau^2 + [Z]^T*[V]*[Z] - 2[Z]^T*[S]}$, where $[Z]^T$ is the transpose matrix of [Z]
(b) $\boldsymbol{\sigma^2/w_{n+d} + \delta_{n+d,n+d} + \tau^2 - [S]^T*\text{inverse}[V]*[S]}$, where $[S]^T$ is the transpose matrix of [S]

Problem CSRP-51. In Klugman's three-level model, the following information is given:
• ξ is the mean for the group, which is stable over time and drawn from a distribution with mean μ and variance τ^2.
• The means for the observations at times 1 through n are $(\theta_1, \ldots, \theta_n)$, drawn from a multivariate distribution where each value has a mean of ξ.
• For any t and s among the values 1, 2, …, n, n+d, let $Cov(\theta_t, \theta_s) = \delta_{t,s}$.
• The value for the observation X_t is drawn for a distribution with mean θ_t and variance σ^2/w_t, where w_t is a known weight.

In this model, the solution for the vector of credibility weights [Z] can be arrived at via the matrix equation [Z] = inverse[[D] + [Σ] + [T]]*[S]. (Klugman, p. 12)

What two elements of this equation are modified in a Bühlmann-Straub model? (Klugman, p. 13)

Solution CSRP-51. In a Bühlmann-Straub model:
1. The matrix [Σ] becomes the zero matrix.
2. All elements in matrix [S] become equal to τ^2.

Problem CSRP-52.
(a) Why does Klugman state on page 13 that the three-level he presented has far too many parameters to be useful?
(b) What two simplifications does Klugman introduce to resolve this problem?

Solution CSRP-52.
(a) There would never be enough data points to estimate all of this model's parameters.
(b) The two simplifications are the following:
1. All groups have the same parameters.
2. The covariance terms depend only on the difference between the subscripts.

Problem CSRP-53. ***Fill in the blank*** (Klugman, p. 13): The simplifying assumption that the covariance terms depend only on the difference between the subscripts is consistent with the __________ requirement when performing time-series analysis.

Solution CSRP-53. The simplifying assumption that the covariance terms depend only on the difference between the subscripts is consistent with the **stationarity** requirement when performing time-series analysis.

Problem CSRP-54. How will the following matrix from Klugman's three-level model be changed under the simplifying assumptions mentioned by Klugman on page 13?

[Σ] =

$\delta_{1,1}$	$\delta_{1,2}$	…	$\delta_{1,n}$
$\delta_{2,1}$	$\delta_{2,2}$	…	$\delta_{2,n}$
…	…	…	…
$\delta_{n,1}$	$\delta_{n,2}$	…	$\delta_{n,n}$

Solution CSRP-54. The second simplifying assumption is that the covariance terms depend only on the difference between the subscripts. The differences are 0 on the main diagonal and increase by 1 as one moves out by one diagonal. The result is as follows:

[Σ] =

δ_0	δ_1	…	δ_{n-1}
δ_1	δ_0	…	δ_{n-2}
…	…	…	…
δ_{n-1}	δ_{n-2}	…	δ_0

Problem CSRP-55. How will the following matrix from Klugman's three-level model be changed under the simplifying assumptions mentioned by Klugman on page 13?

[S] =

$\delta_{1,n+d} + \tau^2$
$\delta_{2,n+d} + \tau^2$
…
$\delta_{n,n+d} + \tau^2$

Solution CSRP-55. The second simplifying assumption is that the covariance terms depend only on the difference between the subscripts. Thus, $\delta_{n,n+d}$ becomes δ_d, and $\delta_{1,n+d}$ becomes δ_{n+d-1}.

The resulting matrix is [S] =

$\delta_{n+d-1} + \tau^2$
$\delta_{n+d-2} + \tau^2$
…
$\delta_d + \tau^2$

Problem CSRP-56.
(a) How many total parameters does Klugman's simplified version of the three-level model have? Identify what these parameters are.
(b) Why, according to Klugman, is this model still overspecified? (Klugman, p. 13)

Solution CSRP-56.
(a) The total number of parameters is **n + d + 2.**
The parameters are as follows: σ^2, τ^2, and (n+d) values of δ (ranging from δ_0 to δ_{n+d-1}).
(b) This model is overspecified because there can be at most (n+d+1) free parameters, and this model has one parameter more than that.

Problem CSRP-57. In Klugman's random effects model formulation on page 14, it is given that, for group i, the $[X_i]$, the vector of observations $X_{i,1}, \ldots, X_{i,n}$ is the sum of three vectors $[\mu]$, $[\theta_i]$, and $[\varepsilon_i]$: $[X_i] = [\mu] + [\theta_i] + [\varepsilon_i]$.

(a) Define the vector $[\mu]$.
(b) Define the vector $[\theta_i]$.
(c) Define the vector $[\varepsilon_i]$.

Solution CSRP-57.
(a) $[\mu]$ is a vector of n values of the mean μ.
(b) $[\theta_i]$ is a vector of n observations $\theta_{i,1}, \ldots, \theta_{i,n}$ which each follow a Normal distribution with mean 0 and variance drawn from the matrix $[\Sigma] + [T]$.
(c) $[\varepsilon_i]$ is a vector of n observations $\varepsilon_{i,1}, \ldots, \varepsilon_{i,n}$ which each follow a Normal distribution with mean 0 and variance drawn from the matrix $[D_i]$, the diagonal matrix of σ^2/w_1 through σ^2/w_n for group i.

Problem CSRP-58. For a random effects model you are given the following:
• $[X_i]$, the vector of observations $X_{i,1}, \ldots, X_{i,n}$ is the sum of three vectors $[\mu]$, $[\theta_i]$, and $[\varepsilon_i]$: $[X_i] = [\mu] + [\theta_i] + [\varepsilon_i]$.
• $[\mu]$ is a vector of n values of the mean μ.
• $[\theta_i]$ is a vector of n observations $\theta_{i,1}, \ldots, \theta_{i,n}$ which each follow a Normal distribution with mean 0 and variance drawn from the matrix $[\Sigma] + [T]$.
• $[\varepsilon_i]$ is a vector of n observations $\varepsilon_{i,1}, \ldots, \varepsilon_{i,n}$ which each follow a Normal distribution with mean 0 and variance drawn from the matrix $[D_i]$, the diagonal matrix of σ^2/w_1 through σ^2/w_n for group i.

Let $[V_i] = [\Sigma] + [T] + [D_i]$.
Let the "sum" function with respect to a matrix or vector be the function that adds all the elements of the given matrix or vector.

(a) Express the estimator μ^ for μ in terms of $[V_i]$ and $[X_i]$.
(b) Express the loglikelihood function ln L in terms μ^ (including the vector [μ^] containing n values of μ^), $[V_i]$, and $[X_i]$. (Klugman, p. 15)

Solution CSRP-58.
(a) μ^ = $_{i=1}{}^{k}\Sigma$[sum(inverse[V_i]*[X_i])]/ $_{i=1}{}^{k}\Sigma$[sum(inverse[V_i])]
(b) ln L = -0.5*($_{i=1}{}^{k}\Sigma$[ln(det[V_i])] + ln($_{i=1}{}^{k}\Sigma$[sum(inverse[V_i])]) + $_{i=1}{}^{k}\Sigma$[([X_i]-[μ^])^T*inverse[V_i]*([X_i]-[μ^])])].

Problem CSRP-59. You are given that, in a model that combines time series and credibility, the vector [Z] of the credibility weights Z_1, …, Z_n for the observations X_1, …, X_n can be obtained as the solution to the matrix equation [Z] = inverse[V]*[S], as expressed in Solution CSRP-48.

There is one more weight Z_0, which is a standalone constant. Express a formula for calculating Z_0 using the weights Z_1, …, Z_n and the mean μ. (Klugman, p. 15)

Solution CSRP-59.
Z_0 = (1- $_{t=1}{}^{n}\Sigma$(Z_t))*μ.

Problem CSRP-60. Klugman comments on page 15 that the formulas
[Z] = inverse[[D] + [Σ] + [T]]*[S] = inverse[V]*[S] and Z_0 = (1- $_{t=1}{}^{n}\Sigma$(Z_t))*μ
depend on two knowledge items. What are they?

Solution CSRP-60. These formulas depend on
1. Knowing the correlation structure; and
2. Having estimates of the mean and the parameters that define the correlation structure. (These estimates can be obtained via REML.)
(Klugman, p. 15)

Problem CSRP-61. According to Klugman (p. 15), what is the challenge when using time-series models when applied to actuarial analyses? What is the implication of that challenge?

Solution CSRP-61. The challenge is that, when there are limited data, as in many actuarial analyses, many ARIMA models will fit well, and various correlation structures would make sense. The implication of this is that judgment would be required in selecting a model with reasonable weights.

Problem CSRP-62. *Fill in the blanks* (Klugman, p. 15): Mahler separately poses a ________ and a __________.

Solution CSRP-62. Mahler separately poses a **correlation structure** and a **pattern of weights**. (Klugman, p. 15)

Problem CSRP-63. Klugman (p. 15) discusses Mahler's approach to the selection of weights to minimize the least-squares criterion $\sigma^2/w_{n+d} + \delta_{n+d,n+d} + \tau^2 + [Z]^T*[V]*[Z] - 2[Z]^T*[S]$.

Fill in the blanks: The weights in this approach are constrained to follow the ________. This will lead to a suboptimal result with regard to the ___________, but has the advantage of producing weights that are __________.

Solution CSRP-63. The weights in this approach are constrained to follow the **set pattern**. This will lead to a suboptimal result with regard to the **least-squares criterion**, but has the advantage of producing weights that are **easier to explain**. (Klugman, p. 15)

Problem CSRP-64. In an AR(1) model where the lag autocorrelation coefficient is ρ, what equation holds for parameter δ_g for the covariance between two observations where g is the number of periods separating the two observations? (Klugman, p. 16)

Solution CSRP-64. $\boldsymbol{\delta_g = \delta*\rho^g}$.

Problem CSRP-65. You are given a covariance matrix for 5 years of data as follows:

δ	$\delta\rho$	$\delta\rho^2$	$\delta\rho^3$	$\delta\rho^4$
$\delta\rho$	δ	$\delta\rho$	$\delta\rho^2$	$\delta\rho^3$
$\delta\rho^2$	$\delta\rho$	δ	$\delta\rho$	$\delta\rho^2$
$\delta\rho^3$	$\delta\rho^2$	$\delta\rho$	δ	$\delta\rho$
$\delta\rho^4$	$\delta\rho^3$	$\delta\rho^2$	$\delta\rho$	δ

What time-series model does this matrix represent? How is this evident? (Klugman, p. 16)

Solution CSRP-65. This matrix represents an **AR(1) time-series model**, because the relationship $\boldsymbol{\delta_g = \delta*\rho^g}$ holds for the covariances on each diagonal.

Problem CSRP-66. ***Fill in the blanks*** (Klugman, p. 16): In Mahler's approach, after having posited a correlation structure and estimated the required variances and correlations, a(n) _________ structure can be imposed. This can be done by minimizing the ___________ formula or accepting a suboptimal result that has a(n) _________ pattern.

Solution CSRP-66. In Mahler's approach, after having posited a correlation structure and estimated the required variances and correlations, a **credibility** structure can be imposed. This can be done by minimizing the **mean squared error** formula or accepting a suboptimal result that has an **AR(1)** pattern. (Klugman, p. 16)

Problem CSRP-67. You are given that, in a model that combines time series and credibility, the vector [Z] of the credibility weights $Z_1, \ldots, Z_n$ for the observations $X_1, \ldots, X_n$ can be obtained as the solution to the matrix equation [Z] = inverse[V]*[S], as expressed in Solution CSRP-48.

Klugman (p. 16) discusses an approach for applying an AR(1) pattern to this model, such that the mean square error is minimized, subject to certain constraints in terms of constants a and b, where a and b are each ≥ 0. What are these constraints? What is an inequality that arises, involving both a and b?

Solution CSRP-67.
The constraints are $\boldsymbol{Z_1 = Z_2 = \ldots = Z_{n-1} = a, Z_n = b}$.
The inequality that arises is $\boldsymbol{(n-1)*a + b \leq 1}$.

Problem CSRP-68.
(a) When using an MA(1) model, what is true about the covariance parameters δ_0, δ_1 … , δ_{n-1} within the correlation structure?
(b) When the MA(1) model is used with equal sample sizes, what becomes true about the parameters σ^2 and δ_0? (Klugman, p. 17)

Solution CSRP-68.
(a) Only the parameters δ_0 and δ_1 have unique values; the other covariances are zero.
(b) When the sample sizes are equal, $\sigma^2 + \delta_0$ can appear along the diagonal of the covariance matrix as a single parameter and can be estimated as a single quantity via REML.

Problem CSRP-69. Even though the ARIMA(0, 1, 1) model has geometrically decreasing weights, why can it not be extended to a credibility context? (Klugman, p. 17)

Solution CSRP-69. The ARIMA(0, 1, 1) model is applied to the differences between observations, and the expected value of the differences is zero – so the hypothetical means are assumed to be zero and there is no way to model the variation in the hypothetical means, which is necessary for a credibility model.

Problem CSRP-70. You are given that, in a model that combines time series and credibility, the vector [Z] of the credibility weights $Z_1, \dots, Z_n$ for the observations $X_1, \dots, X_n$ can be obtained as the solution to the matrix equation [Z] = inverse[V]*[S], as expressed in Solution CSRP-48.

Klugman (p. 17) describes Mahler's approach for obtaining geometrically decreasing weights, with weight also being placed on the overall mean. This approach involves certain constraints in terms of constants a and b, where $a > 0$ and $0 \leq b \leq 1$. What are these constraints? What is an inequality that arises, involving both a and b?

Solution CSRP-70.
The constraints are the following: $\mathbf{Z_n = a, Z_t = a*b^{n-t}}$**,** where t = 1, 2, …, n – 1.
The inequality is $\mathbf{a*(1-b^n)/(1-b) \leq 1}$.

Problem CSRP-71. ***Fill in the blank*** (Klugman, p. 18): Within the Mahler approach described by Klugman, for any correlation structure, any ________ can be imposed.

Solution CSRP-71. Within the Mahler approach described by Klugman, for any correlation structure, any **pattern of credibility weights** can be imposed.